First order differential equation

Solve the following equation: $y' + y = xy^3$

Solution

This is a Bernoulli differential equation. We divide the expression by y^3:

$$y'y^{-3} + y^{-2} = x$$

Let $z = y^{-2}$, then $z' = -2y^{-3}y'$, so $\frac{z'}{-2} = y^{-3}y'$, and the equation becomes:

$$\frac{-z'}{2} + z = x$$

$$z' - 2z = -2x$$

Which is linear; we propose the substitution $z = uv$, with $z' = u'v + v'u$:

$$u'v + v'u - 2(uv) = -2x$$

$$v(u' - 2u) + v'u = -2x$$

We solve separately:

$$u' - 2u = 0$$

$$v'u = -2x$$

From the first equation:

$$\frac{du}{dx} - 2u = 0$$

$$du = 2u\,dx$$

$$\frac{du}{u} = 2dx$$

$$\ln(u) = 2x$$

$$u = e^{2x}$$

Substituting into the second equation:

$$v'e^{2x} = -2x$$

$$dv\,e^{2x} = -2x\,dx$$

$$dv = -2xe^{-2x}dx$$

We solve the integral on both sides:

$$\int -2xe^{-2x}dx$$

We use integration by parts: $\int f\,dg = fg - \int f'g$,

$$f = -2x, \quad dg = e^{-2x}dx$$

$$df = -2dx, \quad g = \frac{e^{-2x}}{-2}$$

Thus,

$$\int -2xe^{-2x}dx = -2x\left(\frac{e^{-2x}}{-2}\right) - \int -2\left(\frac{e^{-2x}}{-2}\right)dx = xe^{-2x} + \frac{e^{-2x}}{2} + C = e^{-2x}\left(x + \frac{1}{2}\right) + C$$

$$v = e^{-2x}\left(x + \frac{1}{2}\right) + C$$

Then recall that $z = uv = e^{2x}\left(e^{-2x}\left(x + \frac{1}{2}\right) + C\right) = x + \frac{1}{2} + Ce^{2x}$. Also, since $z = y^{-2}$, we have:

$$y^{-2} = x + \frac{1}{2} + Ce^{2x}$$

$$y = \left(x + \frac{1}{2} + Ce^{2x}\right)^{-1/2}$$